

Summation by parts = partial summation or, how to get 'em all: $\sum_{k=1}^n k^p$, $p = 0, 1, 2, \dots$, in principle!

Forward differences:

$$\Delta x_k := x_{k+1} - x_k$$

Collapsing sum:

$$\begin{aligned} \sum_0^n \Delta x_k &= (x_1 - x_0) + (x_2 - x_1) + \dots + (x_{n+1} - x_n) \\ &= x_{n+1} - x_0 \\ &=: x_k \Big|_0^{n+1}. \end{aligned}$$

$$\Delta u_k v_k = u_{k+1} v_{k+1} - u_k v_k$$

$$\begin{aligned} &= (u_{k+1} - u_k) v_{k+1} + \\ &\quad + u_k (v_{k+1} - v_k) \end{aligned}$$

$$= u_k \Delta v_k + v_{k+1} \Delta u_k$$

$$\sum_{k=0}^n u_k \Delta v_k = u_k v_k \Big|_0^{n+1} - \sum_{k=0}^n v_{k+1} \Delta u_k$$

↑

NB the shift of index.

We could also start the sum at $k=1$, or $k=85$, all three places.

⑥ We know $\sum_1^n k = n$, $n=0, 1, 2, \dots$.

① What is

$$\sum_1^n k = \sum_1^n u_k \Delta v_k ?$$

We take

$$u_k = k \text{ so } \Delta u_k = 1,$$

$$v_k = k \text{ so } \Delta v_k = 1.$$

Thus

$$\begin{aligned} \sum_1^n k &= k^2 \Big|_1^{n+1} - \sum_1^n (k+1) \\ &= (n+1)^2 - 1 - \sum_1^n k - \sum_1^n 1 \end{aligned}$$

that is

$$\begin{aligned} 2 \sum_1^n k &= n^2 + kn + k - k - k \\ &= n(n+1), \end{aligned}$$

$$\sum_1^n k = \frac{n(n+1)}{2}, \quad n=0, 1, 2, \dots$$

② What is

$$\sum_1^n k^2 = \sum_1^n u_k \Delta v_k ?$$

We take

$$\begin{aligned} u_k = k^2 \text{ so } \Delta u_k &= (k+1)^2 - k^2 \\ &= 2k+1 \end{aligned}$$

$$v_k = k \text{ so } \Delta v_k = 1.$$

Thus

$$\begin{aligned}\sum_1^n k^2 &= k^3 \Big|_1^{n+1} - \sum_1^n (k+1)(2k+1) \\ &= (n+1)^3 - 1 - 2 \sum_1^n k^2 \\ &\quad - 3 \sum_1^n k - \sum_1^n 1.\end{aligned}$$

that is

$$\begin{aligned}3 \sum_1^n k^2 &= n^3 + 3n^2 + 3n + 1 - n - 1 \\ &\quad - \frac{3}{2} n(n+1) \\ &= n^3 + \frac{3}{2} n^2 + \frac{1}{2} n \\ &= n \left(n^2 + \frac{3}{2} n + \frac{1}{2} \right) \\ &= n \left(n + \frac{1}{2} \right) (n+1),\end{aligned}$$

so

$$\sum_1^n k^2 = \frac{n(n+\frac{1}{2})(n+1)}{3}.$$

③ What is

$$\sum_1^n k^3 = \sum_1^n u_k \Delta v_k?$$

Take

$$\begin{aligned}u_k &= k^3 \text{ so } \Delta u_k = (k+1)^3 - k^3 \\ &= 3k^2 + 3k + 1\end{aligned}$$

by the binomial theorem (?),

$$v_k = k \text{ so } \Delta v_k = 1.$$

Thus

$$\begin{aligned}
 \sum_1^n k^3 &= k^4 \Big|_1^{n+1} - \\
 &\quad - \sum_1^n (k+1)(3k^2 + 3k + 1) \\
 &= (n+1)^4 - 1 - 3 \sum_1^n k^3 - 6 \sum_1^n k^2 - \\
 &\quad - 4 \sum_1^n k - \sum_1^n 1,
 \end{aligned}$$

so (binomial theorem)

$$\begin{aligned}
 4 \sum_1^n k^3 &= n^4 + 4n^3 + 6n^2 + 4n + \cancel{1} - \cancel{1} \\
 &\quad - 6 \cdot \frac{1}{3} n(n+\frac{1}{2})(n+1) - \\
 &\quad - 4 \cdot \frac{1}{2} n(n+1) - \cancel{n} \\
 &= n^4 + 4n^3 + 6n^2 + 3n - \\
 &\quad - n(2n+1)(n+1) - 2n(n+1) \\
 &= n^4 + 4n^3 + 6n^2 + 3n - \\
 &\quad - n(n+1)(2n+3) \\
 &= n(n^3 + 4n^2 + 6n + \cancel{3} - 2n^2 - 5n - \cancel{3}) \\
 &= n(n^3 + 2n^2 + n) = n^2(n^2 + 2n + 1) \\
 &= n^2(n+1)^2,
 \end{aligned}$$

so

$$\sum_1^n k^3 = \frac{n^2(n+1)^2}{4}$$

④ What is

$$\sum_1^n k^4 = \sum_1^n u_k \Delta v_k?$$

1	2	3	4	5
1	3	6	10	15
1	5	10	10	5

Take

$$u_k = k^4 \text{ so } \Delta u_k = 4k^3 + 6k^2 + \\ + 4k + 1$$

$$v_k = k \text{ so } \Delta v_k = 1$$

Thus

$$\begin{aligned} \sum_1^n k^4 &= k^5 \left| \begin{array}{l} \\ \end{array} \right. - \sum_1^{n+1} (k+1)(4k^3 + 6k^2 + 4k + 1) = \\ &= (n+1)^5 - 1 - 4 \sum_1^n k^4 - 10 \sum_1^n k^3 - 10 \sum_1^n k^2 - \\ &\quad - 5 \sum_1^n k - \sum_1^n 1, \end{aligned}$$

$$\begin{aligned} 5 \sum_1^n k^4 &= n^5 + 5n^4 + 10n^3 + 10n^2 + 5n - \\ &\quad - \frac{5}{2} n^2 (n+1)^2 - \frac{5}{3} n (2n+1)(n+1) \\ &\quad - \frac{5}{2} n (n+1) - \cancel{1} \end{aligned}$$

Does this fifth degree polynomial factor nicely? Clearly n divides it. How about $n+1$, that is, is $n=-1$ a zero of $n^5 + 5n^4 + 10n^3 + 10n^2 + 5n$? Well, since $-1 + 5 - 10 + 10 - 4 = 0$, it is. So $n(n+1)$ divides the polynomial.

$$\begin{aligned} n^5 + 5n^4 + 10n^3 + 10n^2 + 5n &= \\ &= n(n^4 + 5n^3 + 10n^2 + 10n + 4) \end{aligned}$$

$$\begin{array}{r}
 n^3 + 4n^2 + 6n + 4 \\
 n+1 \sqrt{n^4 + 5n^3 + 10n^2 + 10n + 4} \\
 \underline{n^4 + n^3} \\
 4n^3 + 10n^2 \\
 \underline{4n^3 + 4n^2} \\
 6n^2 + 10n \\
 \underline{6n^2 + 6n} \\
 4n + 4 \\
 \underline{4n + 4} \\
 0
 \end{array}$$

So

$$\begin{aligned}
 5 \sum_1^n k^4 &= n(n+1) \left(n^3 + 4n^2 + 6n + 4 - \right. \\
 &\quad \left. - \frac{5}{2}n(n+1) - \frac{5}{3}(2n+1) - \frac{5}{2} \right) \\
 &= n(n+1) \left(n^3 + \frac{3}{2}n^2 + \frac{1}{6}n - \frac{1}{6} \right)
 \end{aligned}$$

Does $n + \frac{1}{2}$ divide it? Well,

$$\begin{aligned}
 (-\frac{1}{2})^3 + \frac{3}{2}(-\frac{1}{2})^2 + \frac{1}{6}(-\frac{1}{2}) - \frac{1}{6} &= -\frac{1}{8} + \frac{3}{8} - \frac{1}{12} - \frac{1}{6} = \\
 &= \frac{1}{4} - \frac{3}{12} = 0, \text{ so } \underline{\text{yes!}}
 \end{aligned}$$

$$\begin{array}{r}
 n^2 + n - \frac{1}{3} \\
 n + \frac{1}{2} \sqrt{n^3 + \frac{3}{2}n^2 + \frac{1}{6}n - \frac{1}{6}} \\
 \underline{n^3 + \frac{1}{2}n^2} \\
 \phantom{n^3 + \frac{1}{2}n^2} n^2 + \frac{1}{6}n \\
 \phantom{n^2 + \frac{1}{6}n} n^2 + \frac{1}{2}n \\
 \underline{\phantom{n^2 + \frac{1}{6}n} n^2 + \frac{1}{2}n} \\
 \phantom{\phantom{n^2 + \frac{1}{6}n} n^2 + \frac{1}{2}n} -\frac{1}{3}n - \frac{1}{6} \\
 \underline{-\frac{1}{3}n - \frac{1}{6}} \\
 \phantom{-\frac{1}{3}n - \frac{1}{6}} 0
 \end{array}$$

One answer:

$$\sum_1^n k^4 = \frac{n(n+\frac{1}{2})(n+1)(n^2+n-\frac{1}{3})}{5}$$

if the algebra's correct!

Perhaps we can see the pattern
if we don't factor the polynomials?

$$\textcircled{0} \quad \sum_1^n 1 = n$$

$$\textcircled{1} \quad 2 \sum_1^n k = n + n^2$$

$$\begin{aligned} \textcircled{2} \quad 3 \sum_1^n k^2 &= (n+1)^3 - 1 - 3 \sum_1^n k - \sum_1^n 1 \\ &= n^3 + 3n^2 + 3n \\ &\quad - \frac{3}{2}n^2 - \frac{3}{2}n \\ &\quad - n \\ &= \frac{1}{2}n + \frac{3}{2}n^2 + n^3 \end{aligned}$$

$$\begin{aligned} \textcircled{3} \quad 4 \sum_1^n k^3 &= (n+1)^4 - 1 - 6 \sum_1^n k^2 - \\ &\quad - 4 \sum_1^n k - \sum_1^n 1 \\ &= n^4 + 4n^3 + 6n^2 + 4n + 1 - 1 \\ &\quad - 2n^3 - 3n^2 - n \\ &\quad - 2n^2 - 2n \\ &\quad - n \\ &= 0n + 1n^2 + 2n^3 + n^4 \end{aligned}$$

$$\begin{aligned} \textcircled{4} \quad \sum_1^n k^4 &= k^5 \Big|_1^{n+1} - \sum_1^n (k+1)(4k^3 + 6k^2 + 4k + 1) \\ &= (n+1)^5 - 1 - 4 \sum_1^n k^4 - 10 \sum_1^n k^3 - \\ &\quad - 10 \sum_1^n k^2 - 5 \sum_1^n k - \sum_1^n 1 \end{aligned}$$

$$\begin{aligned}
 5 \sum_1^n k^4 &= n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1 - x \\
 &\quad - \frac{10}{4} (n^4 + 2n^3 + 1n^2 + 0n) \\
 &\quad - \frac{10}{3} (n^3 + \frac{3}{2}n^2 + \frac{1}{2}n) \\
 &\quad - \frac{5}{2} (n^2 + n) \\
 &\quad - 1 (n) \\
 &= n^5 + \frac{5}{2}n^4 + \frac{5}{3}n^3 + 0n^2 - \frac{1}{6}n \\
 &= -\frac{1}{6}n + 0n^2 + \frac{5}{3}n^3 + \frac{5}{2}n^4 + 1 \cdot n^5
 \end{aligned}$$

$$\textcircled{5} \quad \sum_{k=1}^n k^5 = k^6 \left| \begin{array}{l} \\ \end{array} \right| - \sum_{k=1}^n (k+1)(5k^4 + 10k^3 + 10k^2 + 5k + 1)$$

$$5k^4 + 10k^3 + 10k^2 + 5k + 1$$

14 + 1

$$5k^5 + 10k^4 + 10k^3 + 5k^2 + k$$

$$5k^4 + \underline{10k^3 + 10k^2 + 5k + 1}$$

$$5k^5 + 15k^4 + 20k^3 + 15k^2 + 6k + 1$$

$$\begin{array}{cccc} & 1 & & \\ & | & & \\ & 1 & 2 & 1 \\ & | & 3 & 3 & 1 \end{array}$$

$$\begin{array}{r} 1 \ 4 \ 6 \ 4 \ 1 \\ 5 \ 10 \ 10 \ 5 \ 1 \end{array}$$

1 5 10 10 5 1

1 6 15 20 15 6 1

There is method
to this madness!

It involves

binomial coefficients!

Systematize it!

That is, what's the induction hypothesis?

$$\begin{aligned}
 6 \sum_k k^5 &= (n+1)^6 - 1 - 15 \sum_k k^4 - 20 \sum_k k^3 \\
 &\quad - 15 \sum_k k^2 - 6 \sum_k k - \sum_k 1 \\
 &= n^6 + 6n^5 + 15n^4 + 20n^3 + 15n^2 + 6n + 1 - 1 \\
 &\quad - \frac{15}{5} \left(n^5 + \frac{5}{2}n^4 + \frac{5}{3}n^3 + 0n^2 - \frac{1}{6}n \right) \\
 &\quad - \frac{20}{4} \left(n^4 + 2n^3 + 1n^2 + 0n \right) \\
 &\quad - \frac{15}{3} \left(n^3 + \frac{3}{2}n^2 + \frac{1}{2}n \right) \\
 &\quad - \frac{6}{2} \left(n^2 + n \right) \\
 &\quad - 1 \left(n \right) \\
 &= n^6 + 3n^5 + \frac{5}{2}n^4 + 0n^3 - \frac{1}{2}n^2 + 0n \\
 &= 0n - \frac{1}{2}n^2 + 0n^3 + \frac{5}{2}n^4 + 3n^5 + 1 \cdot 1,
 \end{aligned}$$

⑥ Next row of Pascal triangle

$$1 \ 7 \ \underbrace{21 \ 35 \ 35}_{\text{row}} \ 21 \ 7 \ 1$$

$$\begin{aligned}
 7 \sum_k k^6 &= (n+1)^7 - 1 - 21 \sum_k k^5 - 35 \sum_k k^4 \\
 &\quad - 35 \sum_k k^3 - 21 \sum_k k^2 - 7 \sum_k k \\
 &\quad - 1 \sum_k 1
 \end{aligned}$$

$$\begin{aligned}
 & \sum_1^n k^6 = n^7 + 7n^6 + 21n^5 + 35n^4 + 35n^3 + 21n^2 + 7n \\
 & - \frac{21}{6} \left(n^6 + 3n^5 + \frac{5}{2}n^4 + 0n^3 - \frac{1}{2}n^2 + 0n \right) \\
 & - \frac{35}{5} \left(n^5 + \frac{5}{2}n^4 + \frac{5}{3}n^3 + 0n^2 - \frac{1}{6}n \right) \\
 & - \frac{35}{4} \left(n^4 + 2n^3 + (n^2 + 0n) \right) \\
 & - \frac{21}{3} \left(n^3 + \frac{3}{2}n^2 + \frac{1}{2}n \right) \\
 & - \frac{7}{2} \left(n^2 + n \right) \\
 & - 1 \left(n \right) \\
 = & n^7 + \frac{7}{2}n^6 + \frac{7}{2}n^5 + 0n^4 - \frac{7}{6}n^3 + 0n^2 + \frac{1}{6}n \\
 = & \frac{1}{6}n + 0n^2 - \frac{7}{6}n^3 + 0n^4 + \frac{7}{2}n^5 + \frac{7}{2}n^6 + 1 \cdot n^7 \\
 = & n(n + \frac{1}{2})(n + 1)(n^4 + 2n^3 - n + \frac{1}{3})
 \end{aligned}$$

⋮

It's known that they are all known. But what are they?

"TBC".